Second semestral Examination 2015
B.Math. (Hons.) IIIrd year
Algebraic Number Theory
May 8, 2015
Instructor - B.Sury
Each question carries equal marks
Maximum marks 50

Q 1. Determine with proof the integral closures of the domains:
(i) $\mathbb{Z}[\sqrt{d}]$, for a square-free integer d;
(ii) $\{\sum_{n\geq 0} c_n X^n \in \mathbb{Q}[[X]] : c_1 = 0\}$.

OR

Let K be a number field such that $O_K = \mathbb{Z}[\alpha]$. Let α_i $(1 \leq i \leq n)$ be the conjugates of α, and assume that $\prod_{i<j}(\alpha_i - \alpha_j)^2$ is a square-free integer. Prove that a prime number p which is ramified in O_K must divide the above integer.

Q 2. If α is a root of $X^5 - X - 1 = 0$, and $K = \mathbb{Q}(\alpha)$, show that the square of the ideal $(19, \alpha + 6)$ divides $19O_K$.
Hint: You may prove that Dedekind-Kummer criterion is applicable and apply it.

OR

Let p be a prime of the form $2^n + 1$. If K is the cyclotomic field generated by the p-th roots of unity, determine the number of prime ideals lying over 2 in terms of n.
Hint: You may use without proof the Kummer-Dedekind criterion.

Q 3. Let $A \subset B$ be domains and suppose C is the integral closure of A in B. Let $g, h \in B[X]$ be monic polynomials such that $gh \in C[X]$. Prove that $g, h \in C[X]$.

OR

Determine the class group of $\mathbb{Q}(\sqrt{-21})$.

Q 4. If K is a cubic extension of $\mathbb{Q}$ with discriminant $-d$ where $0 < d < 50$, prove that O_K is a PID.

OR

Show that the fundamental unit of the cyclotomic field $\mathbb{Q}(\zeta)$, generated by a primitive 5-th root of unity ζ, is $-\zeta^2(1+\zeta)$.

Q 5. Assuming that the class group of $\mathbb{Q}(\sqrt{-5})$ has order 2, show that the Diophantine equation $x^3 - y^2 = 5$ has no solutions in integers.

OR

Let p be an odd prime number and $K = \mathbb{Q}(\sqrt{1-4p})$. If K has class number 1, show that $X^2 + X + p$ takes prime values at the consecutive integers $0, 1, \cdots, p-1$.
Hint: Argue by contradiction; if $a^2 + a + p$ is composite, show the existence of a prime divisor $q < p$ and look at qO_K.